A Code Verification Exercise for a Boundary Element Method Based on (Un)structured Grids

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1. Error Estimation with Power Series Expansions

- One of the most general methods for error estimation is based on power series expansions

$$e(\phi) = \phi_i - \phi_{exact} \cong e_o + \alpha h_i^p$$

ϕ_i → Local or functional flow quantity for grid i

ϕ_{exact} → Exact solution of local or functional flow quantity

e_o → Error for cell size zero (supposed to be “0”)

α → Constant

h_i → Typical cell size

p → Observed order of accuracy

2. Geometrical Similarity

- Typical cell size definition, h_i

$$\Lambda = \begin{cases} l \Leftarrow n_{space} = 1 \\ S \Leftarrow n_{space} = 2, \\ V \Leftarrow n_{space} = 3 \end{cases} \quad \Lambda_1 = \frac{\sum_{i=1}^{N_{cells}} \Lambda_i}{N_{cells}}, \quad \Lambda_2 = \sqrt{\frac{\sum_{i=1}^{N_{cells}} \Lambda_i^2}{N_{cells}}}, \quad \Lambda_{\text{mode}}$$

$$\left(\frac{h_i}{h_1}\right)_N = \left(\frac{(N_{cells})_1}{(N_{cells})_i}\right)^{\frac{1}{n_{space}}}, \quad \left(\frac{h_i}{h_1}\right)_1 = \left(\frac{(\Lambda_1)_i}{(\Lambda_1)_1}\right)^{\frac{1}{n_{space}}}$$

$$\left(\frac{h_i}{h_1}\right)_2 = \left(\frac{(\Lambda_2)_i}{(\Lambda_2)_1}\right)^{\frac{1}{n_{space}}}, \quad \left(\frac{h_i}{h_1}\right)_{\text{mode}} = \left(\frac{(\Lambda_{\text{mode}})_i}{(\Lambda_{\text{mode}})_1}\right)^{\frac{1}{n_{space}}}$$

2. Geometrical Similarity (a 1-D example)

- Definition of a typical cell size (single parameter) requires sets of geometrically similar grids

  - Consider a 1-D problem for $0 \leq x \leq 1$

    Geometrically similar grids have a single definition of the dimensionless distance (*s*) as a function of the "dimensionless" node counter (ξ)

$$s = \frac{x - x_{\min}}{x_{\max} - x_{\min}} = x, \quad \xi = \frac{i-1}{N_x - 1}$$

2. Geometrical Similarity (a 1-D example)

$$\frac{d^2\phi}{dx^2} = ax \quad \text{for} \quad 0 < x < 1$$

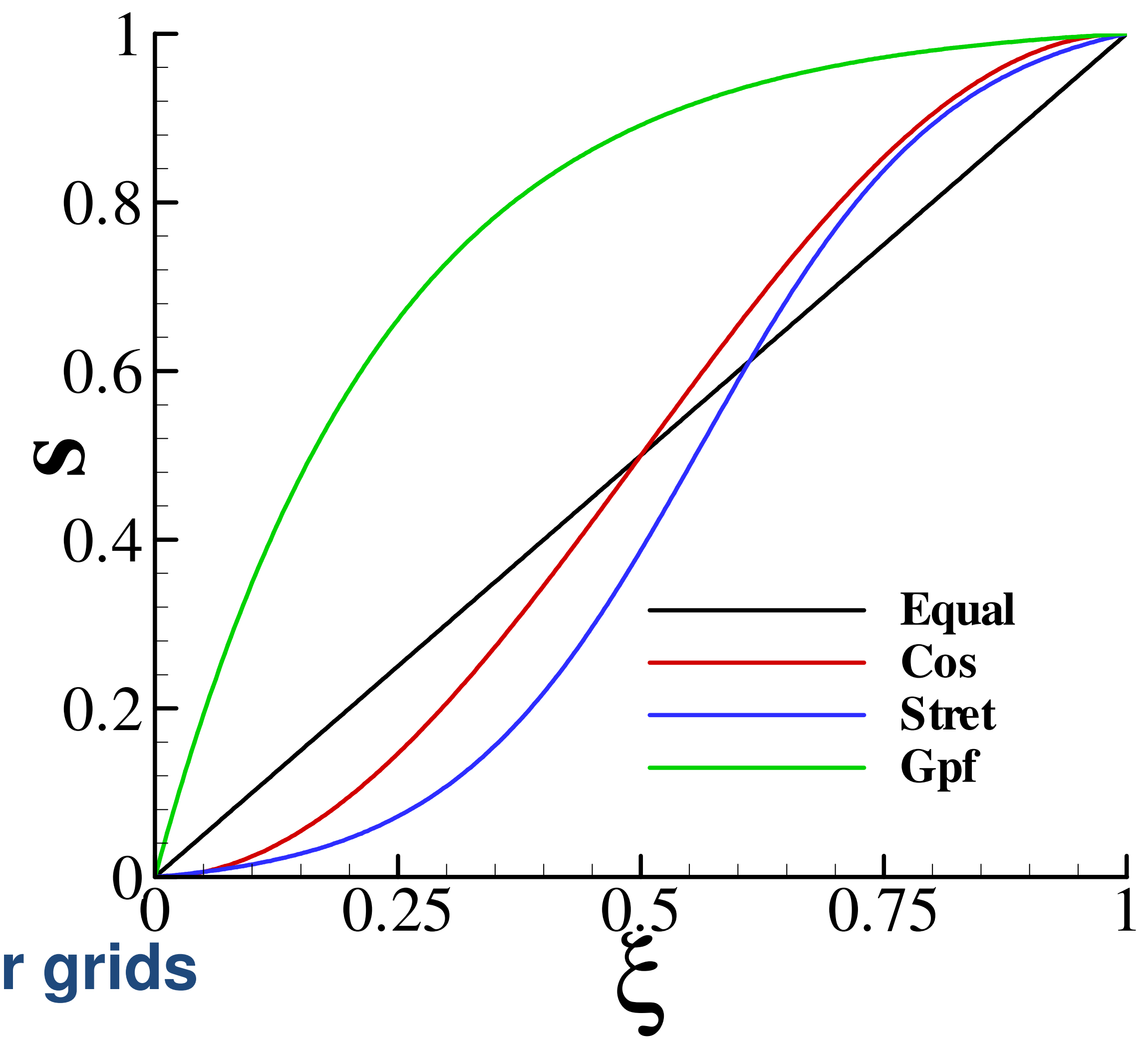
$$\phi(0) = 1 \;\wedge\; \phi(1) = 0$$

$$\phi_{exact} = -\frac{e^{\sqrt{a}}}{e^{-\sqrt{a}} - e^{\sqrt{a}}} e^{-x\sqrt{a}} + \frac{e^{-\sqrt{a}}}{e^{-\sqrt{a}} - e^{\sqrt{a}}} e^{x\sqrt{a}}$$

$$(a = 5)$$

Sets of 31 grids with $5 \leq$ Ncells ≤ 320

2. Geometrical Similarity (a 1-D example)



Similar grids

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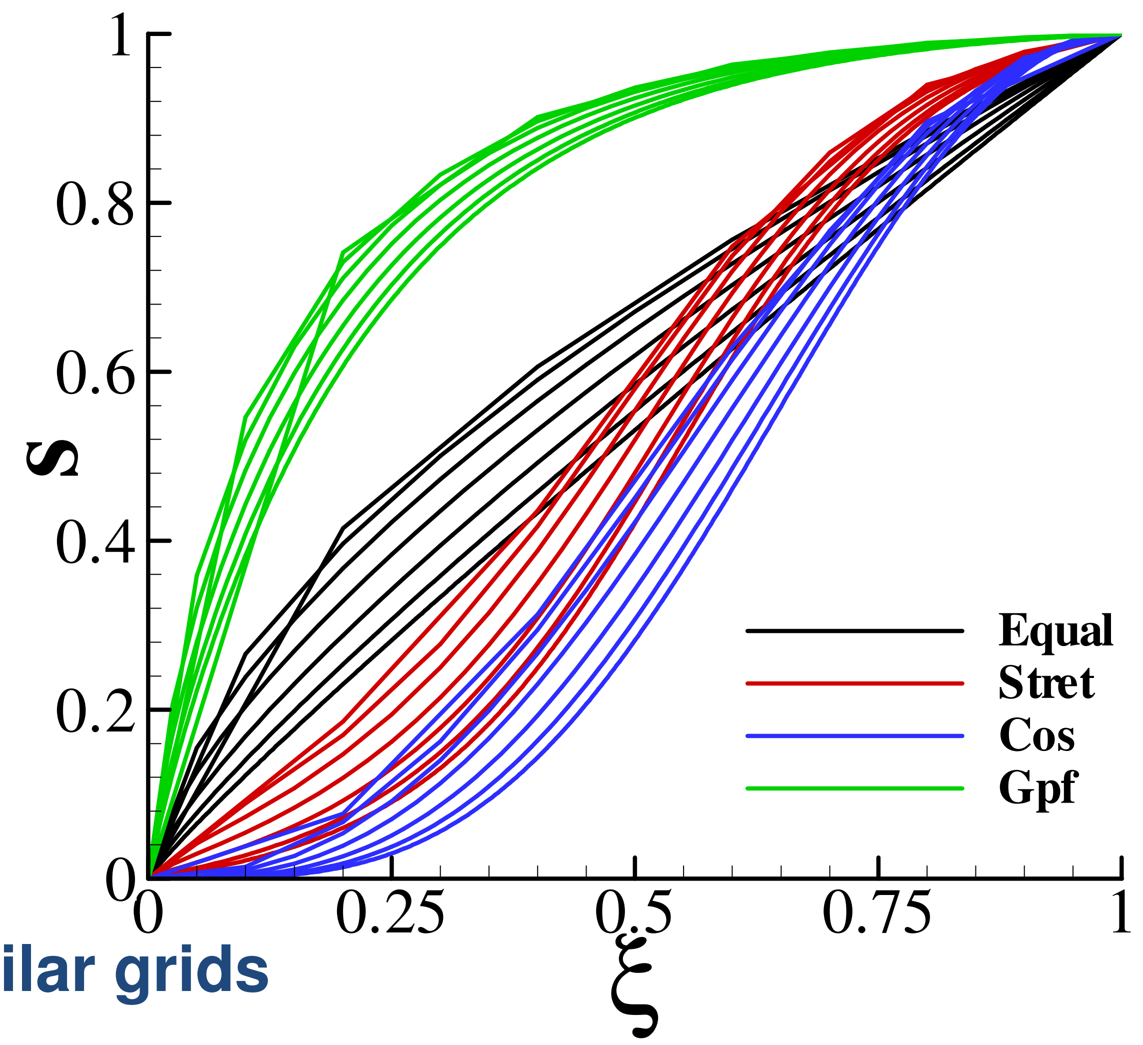

Unsimilar grids

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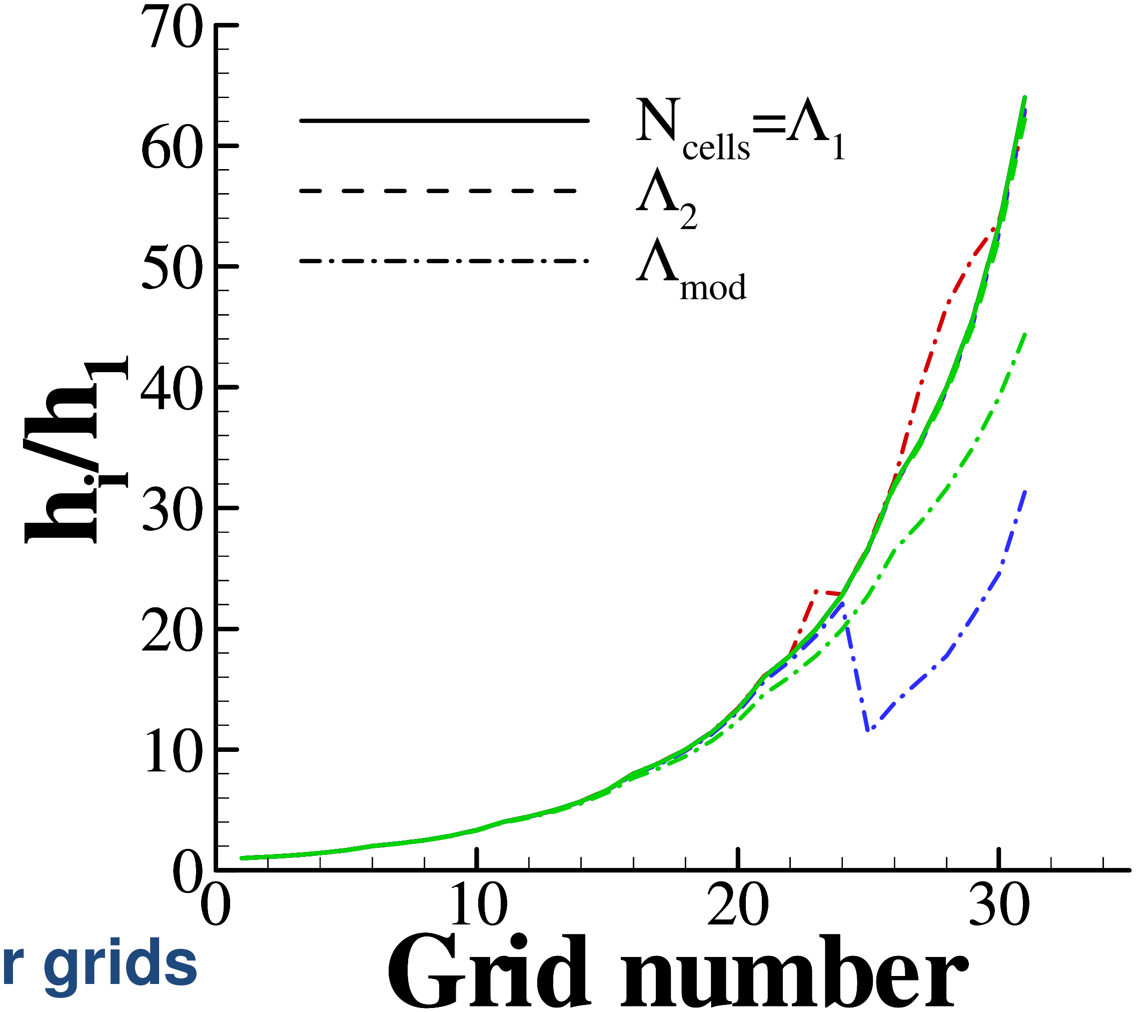

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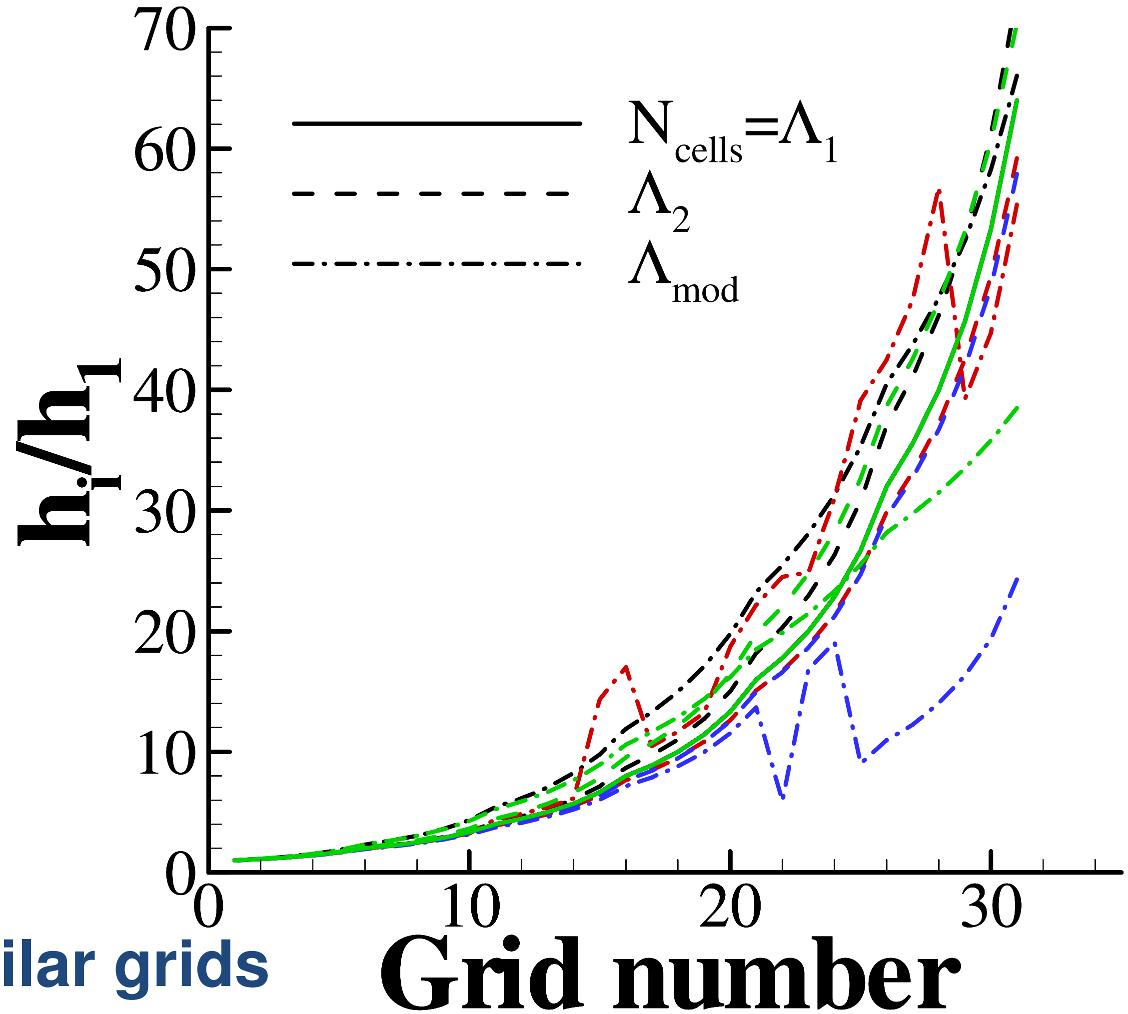

Unsimilar grids

2. Geometrical Similarity (a 1-D example)

- Error norms:

$$L_\infty[e(\phi)] = \max\left(\phi_j - \phi_{exact}\right)$$

$$L_1[e(\phi)] = \frac{\sum_{j=2}^{N_x-1}\left|\phi_j - \phi_{exact}\right|}{N_x - 2}$$

$$L_2[e(\phi)] = \sqrt{\frac{\sum_{j=2}^{N_x-1}\left(\phi_j - \phi_{exact}\right)^2}{N_x - 2}}$$

- $e_o(\phi)$, p and α obtained in the least squares sense from the data of the six finest grids

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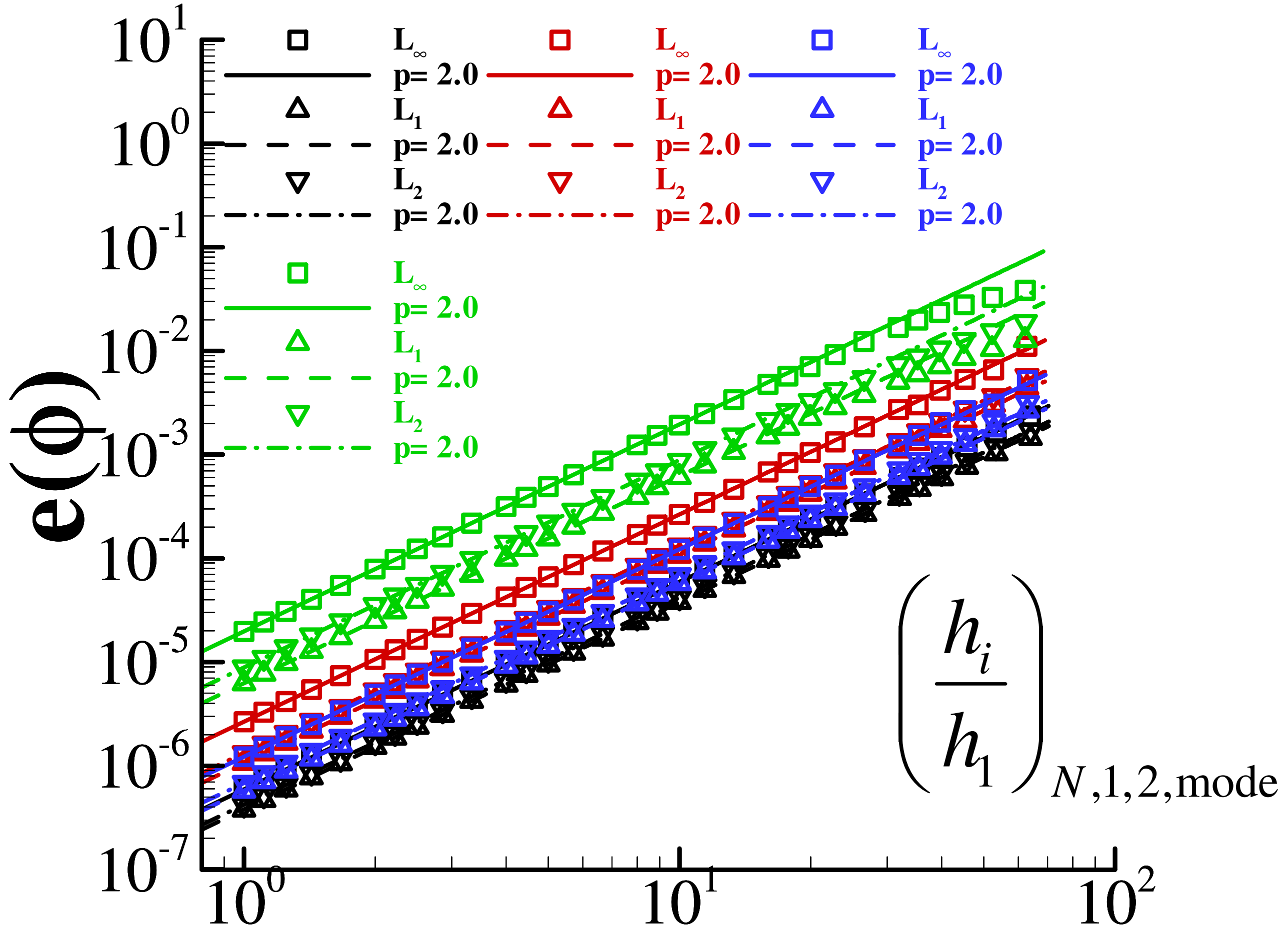

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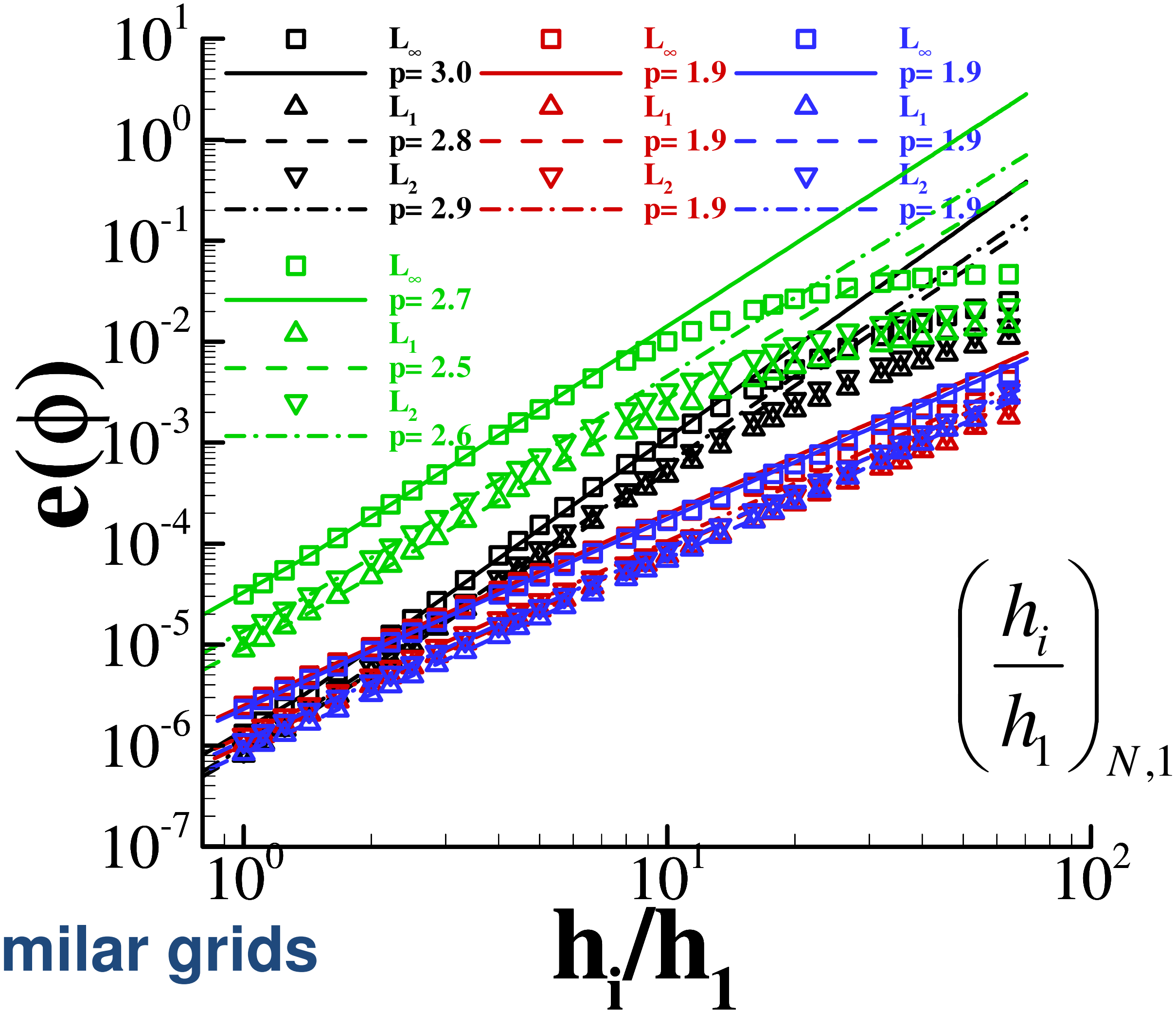

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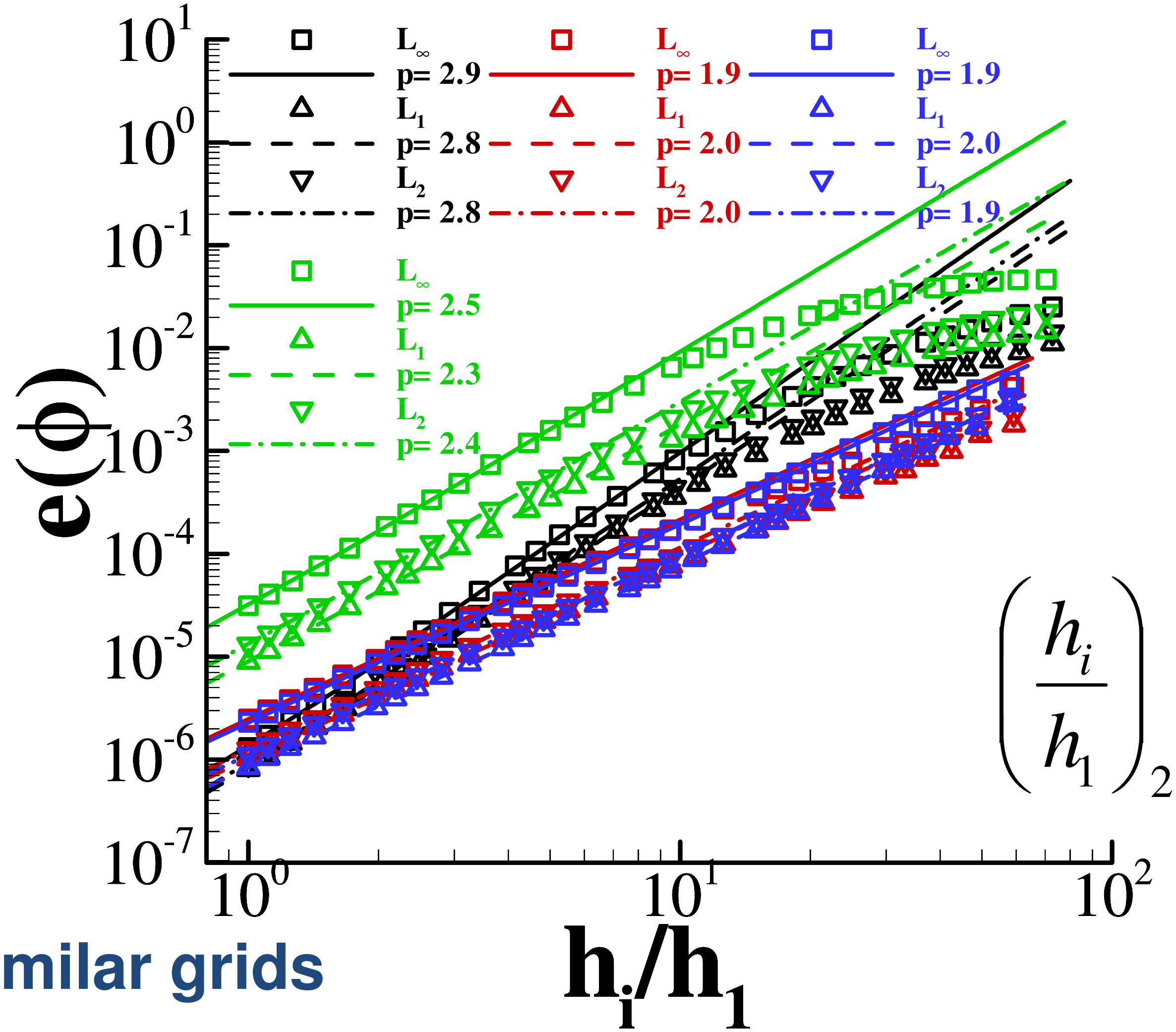

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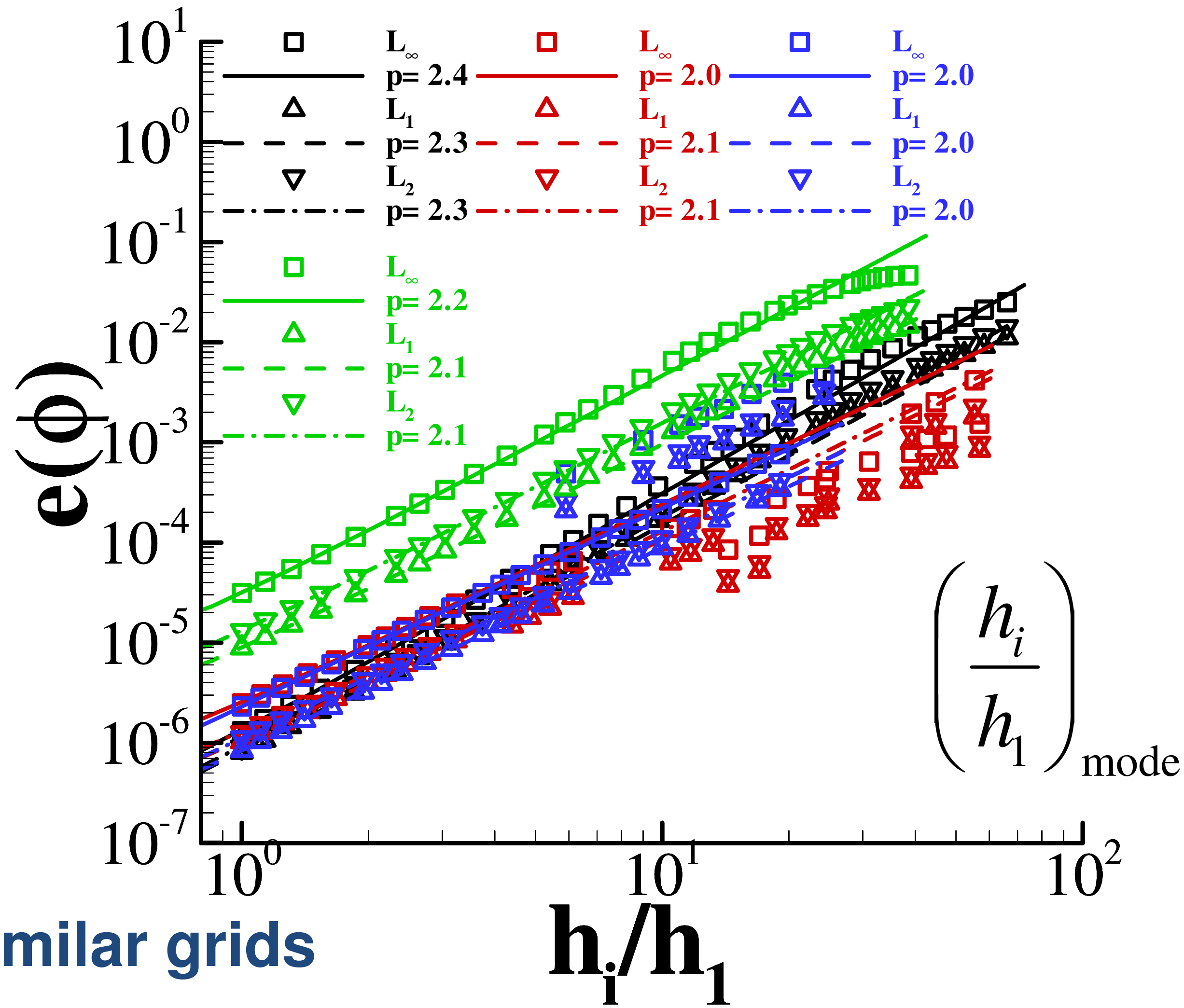


Unsimilar grids

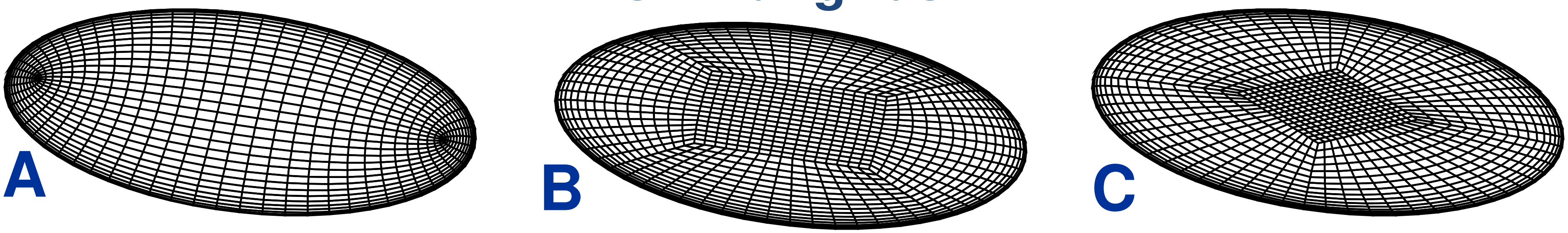
3. Boundary Element Method

- Morino formulation for (non-lifting) potential flow around bodies of arbitrary shape (unknown is the “peturbation” potential (ϕ_p))
- Hyperboloidal panels
- Dipoles and sources distributions with constant strength in each panel
- Uniform undisturbed flow

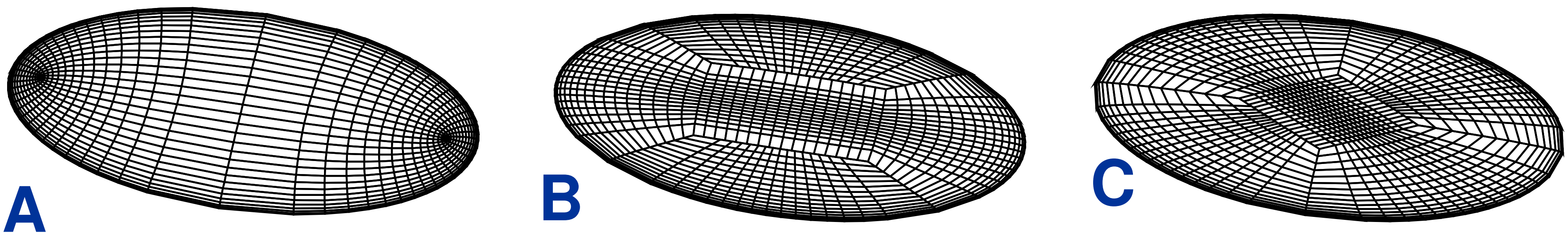
4. Test Case / Grid Sets

$$\left(\frac{x}{1}\right)^2 + \left(\frac{y}{2}\right)^2 + \left(\frac{z}{0.1}\right)^2 = 1$$

Similar grids

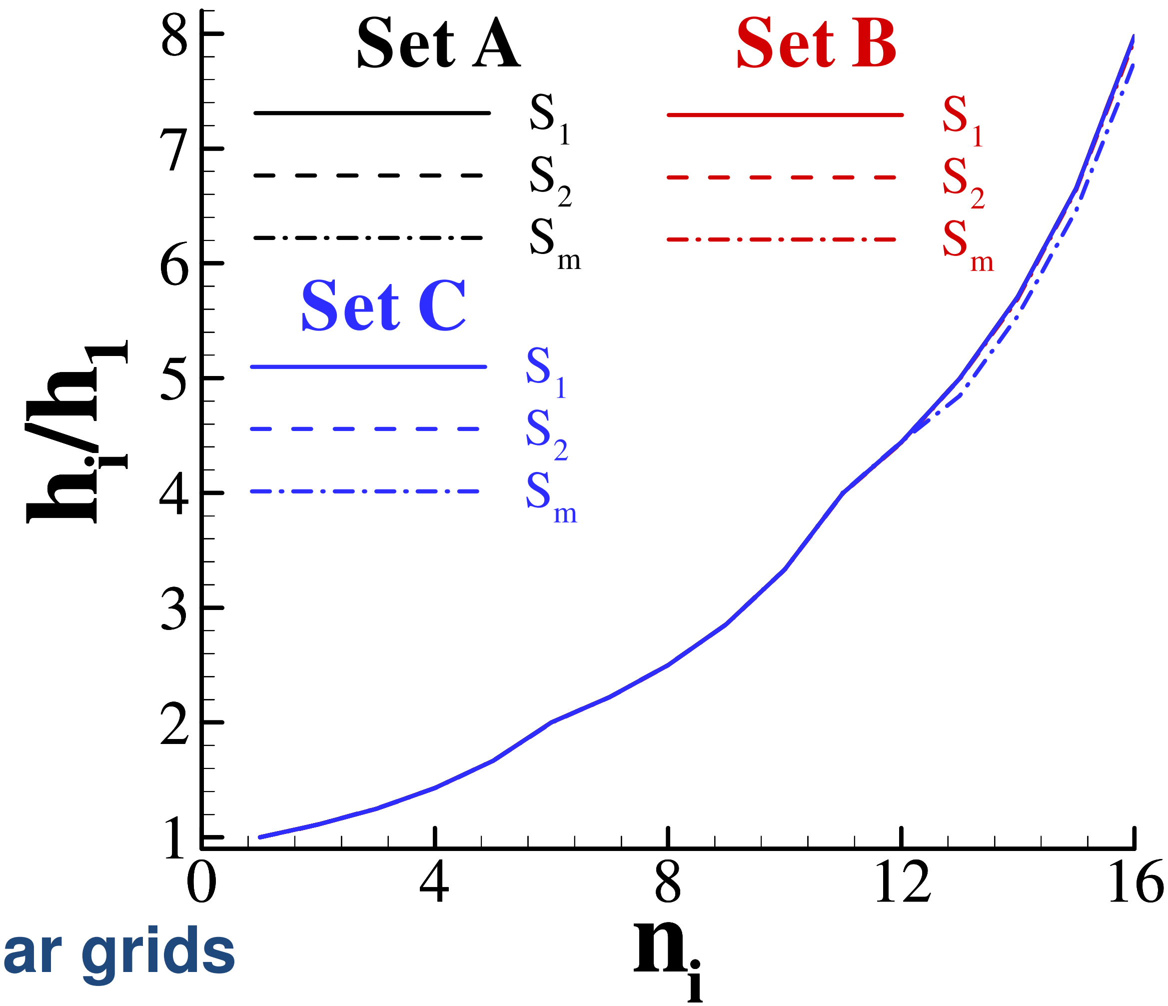


Unsimilar grids



Sets of 16 grids

4. Test Case / Grid Sets



Similar grids

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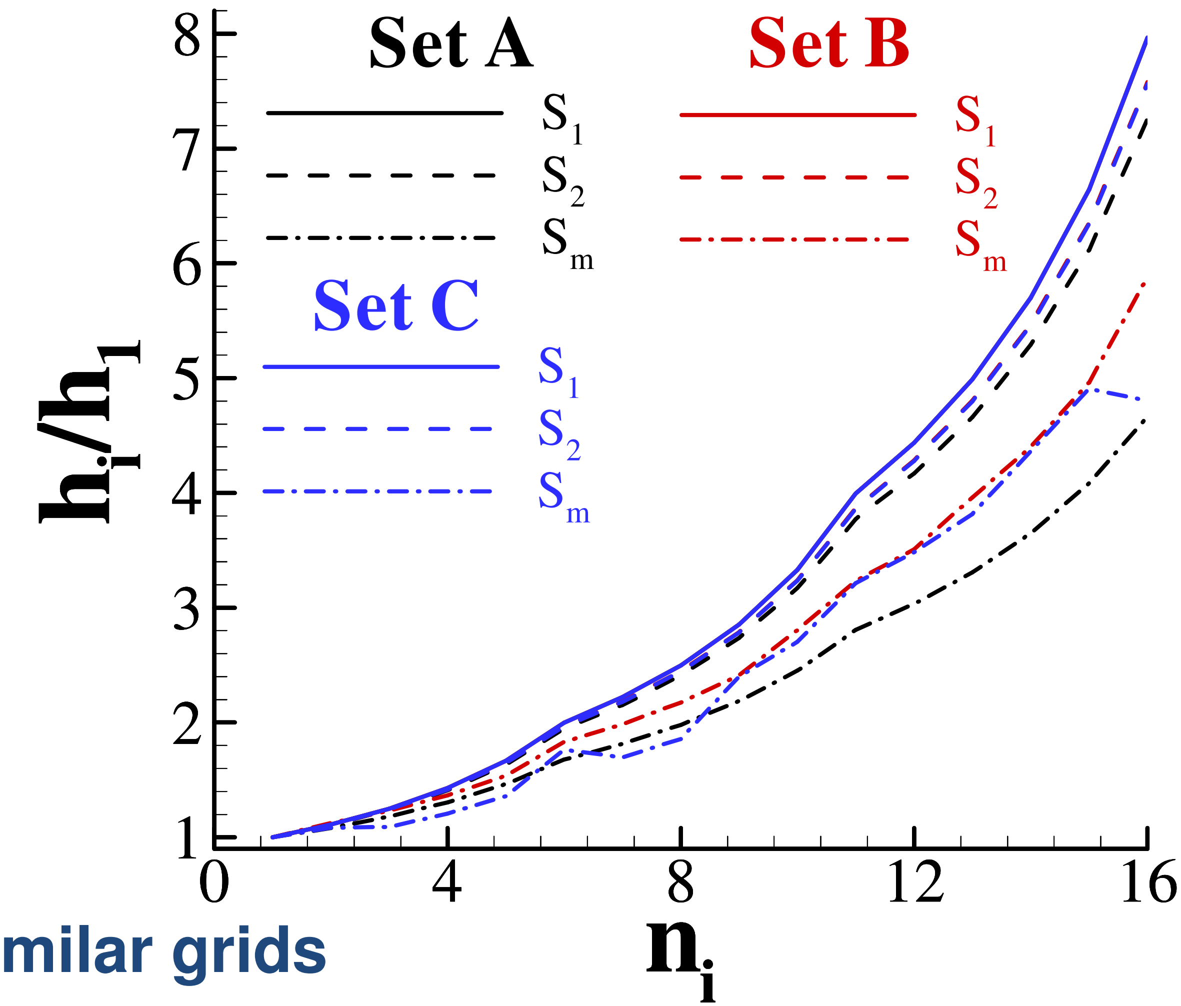


Unsimilar grids

5. Results

- Monitored errors:

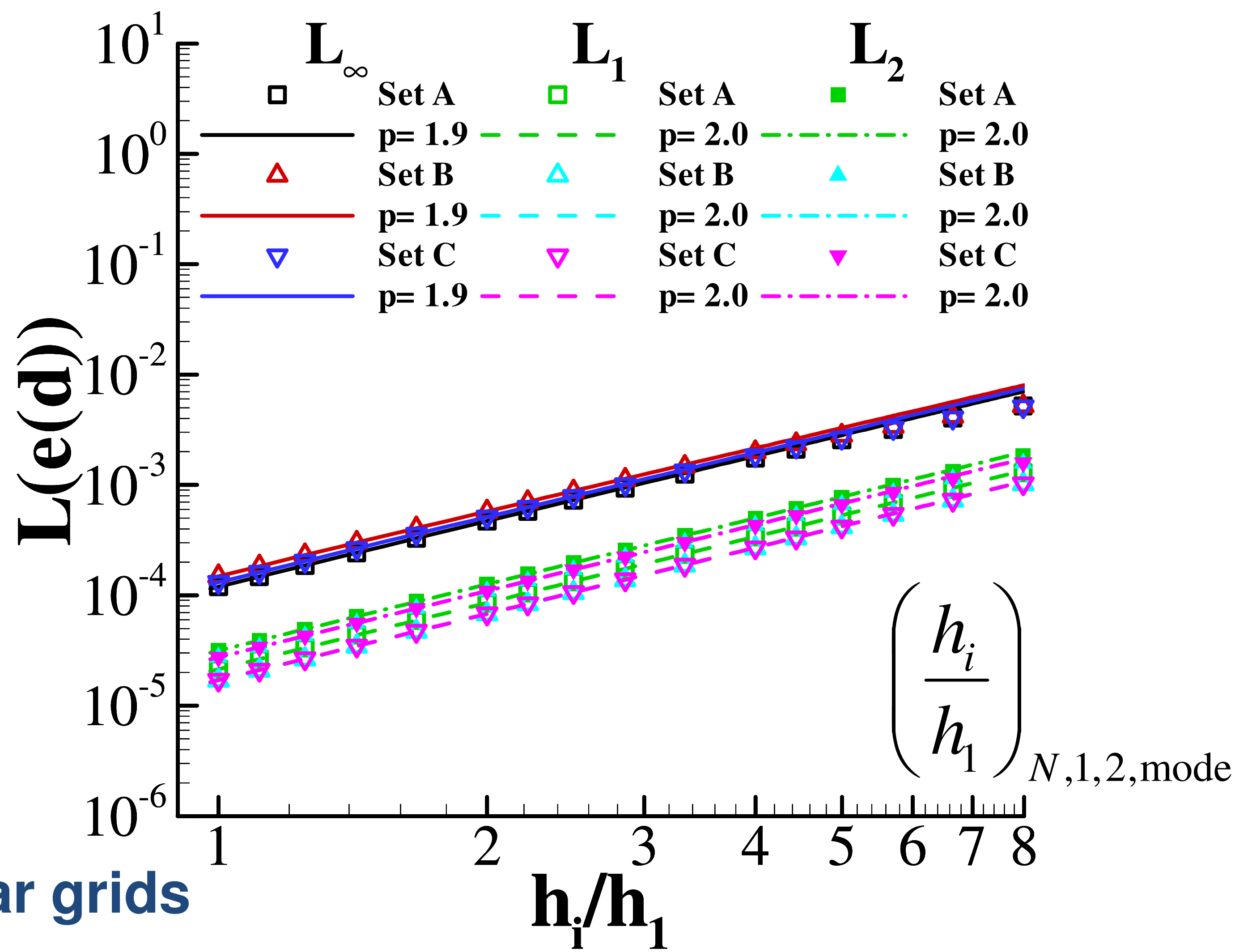
$$e(d)_i = \sqrt{(x_i - x_{exact})^2 + (y_i - y_{exact})^2 + (z_i - z_{exact})^2}$$

$$e(n)_i = \sqrt{((n_x)_i - (n_x)_{exact})^2 + ((n_y)_i - (n_y)_{exact})^2 + ((n_z)_i - (n_z)_{exact})^2}$$

$$e(\phi)_i = \left|(\phi_p)_i - (\phi_p)_{exact}\right|$$

- $e_o(\phi)$, p and α obtained in the least squares sense from the data of the six finest grids

5. Results



Similar grids

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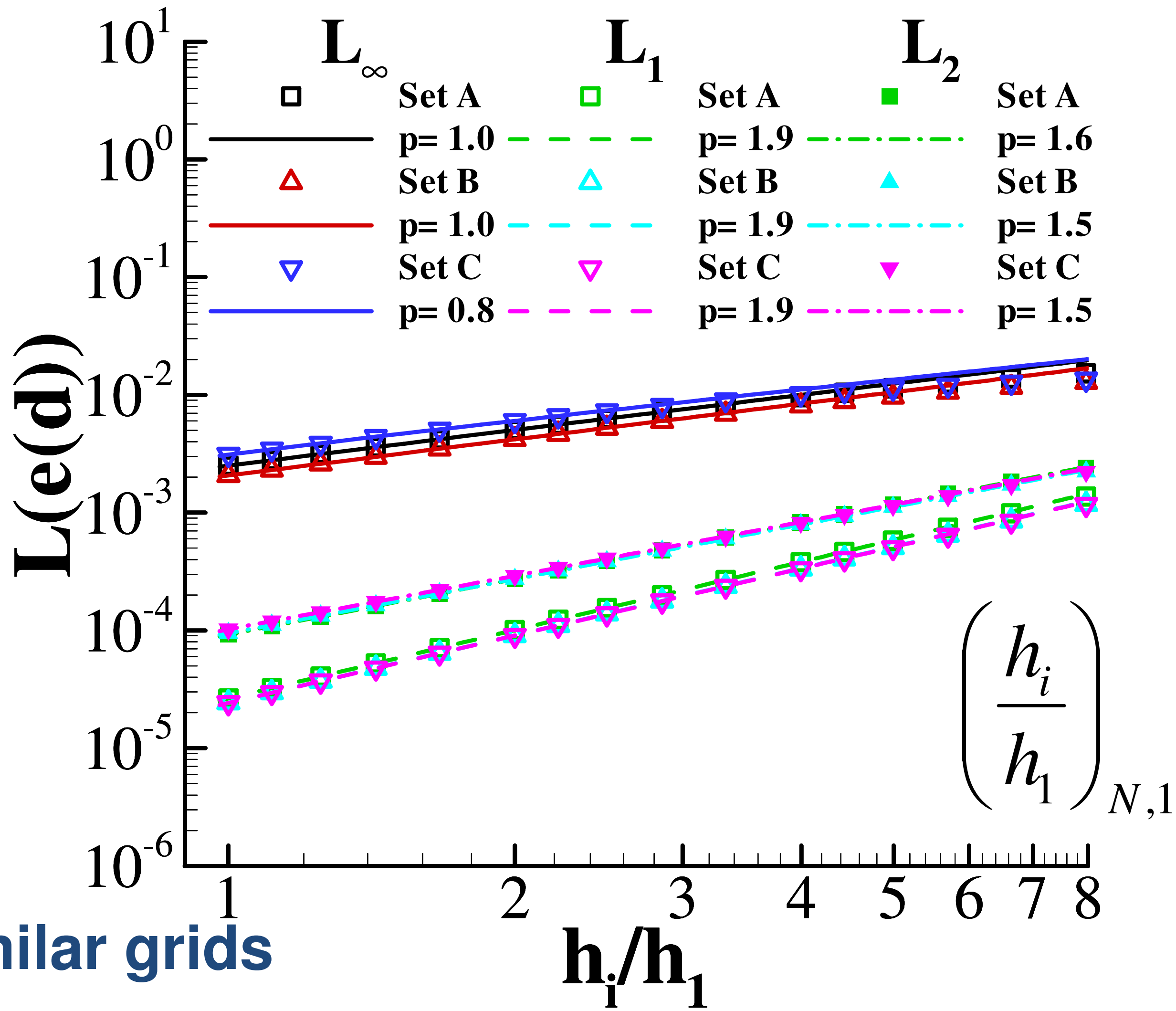

Unsimilar grids

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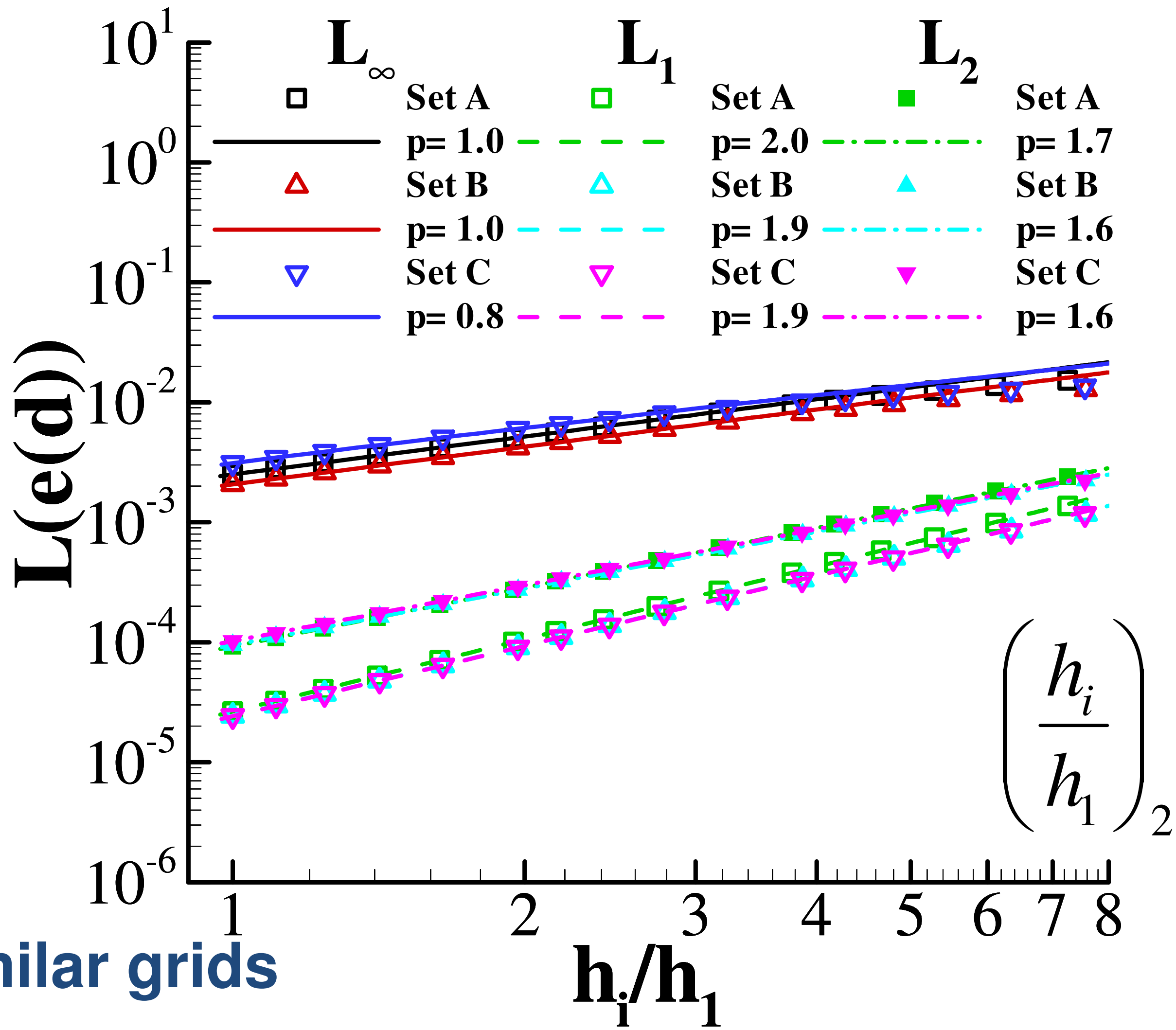

Unsimilar grids

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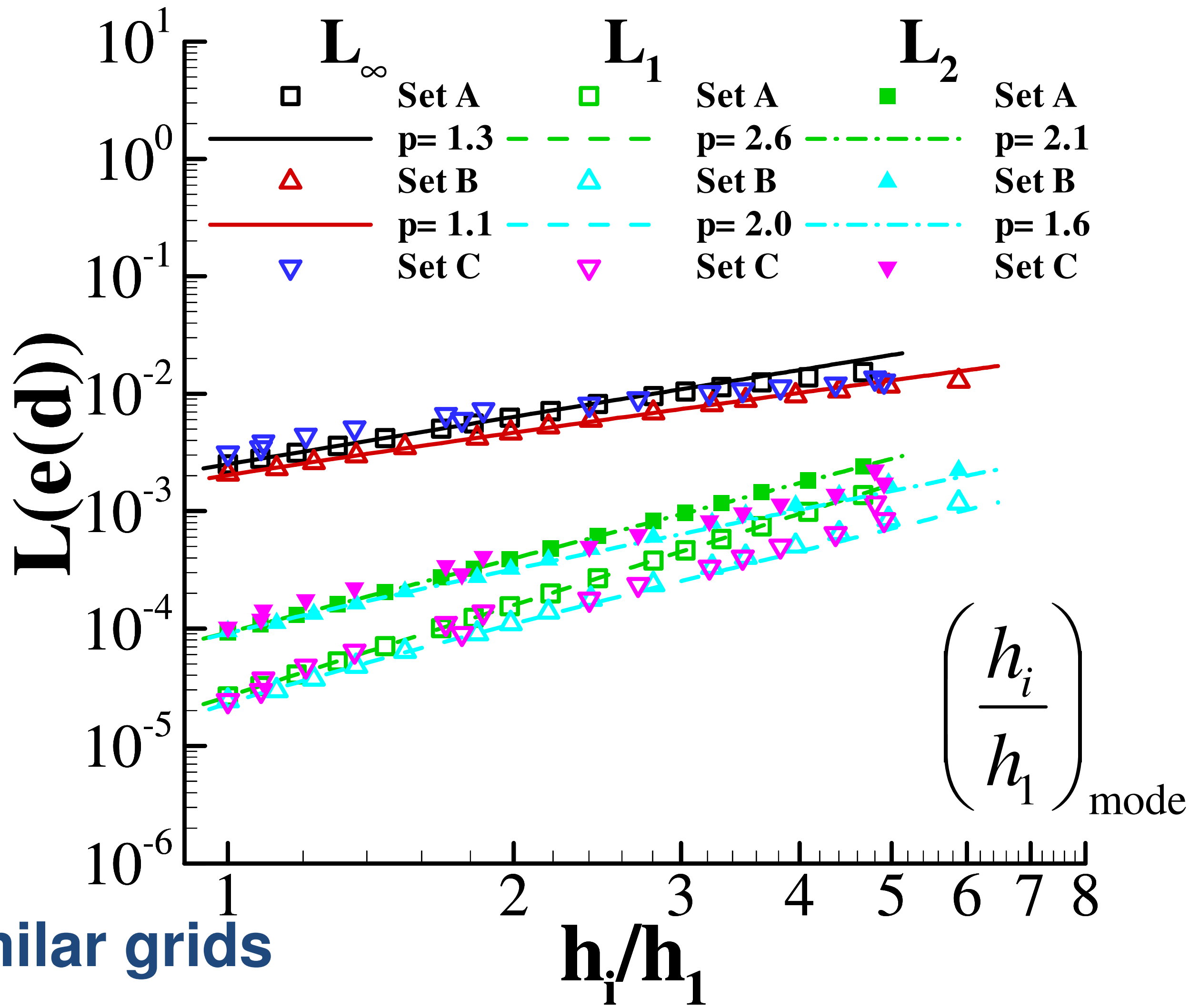

Unsimilar grids

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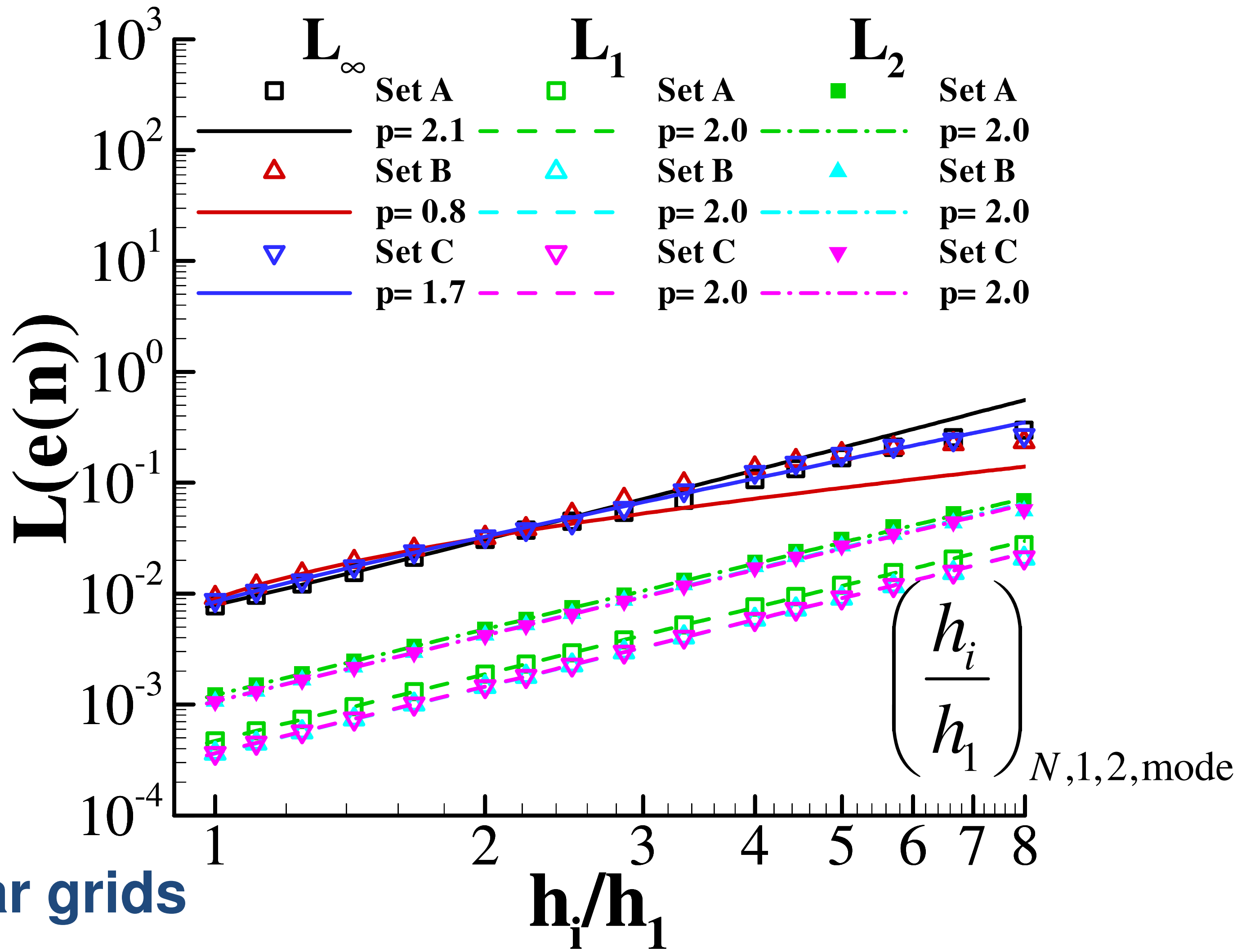

Similar grids

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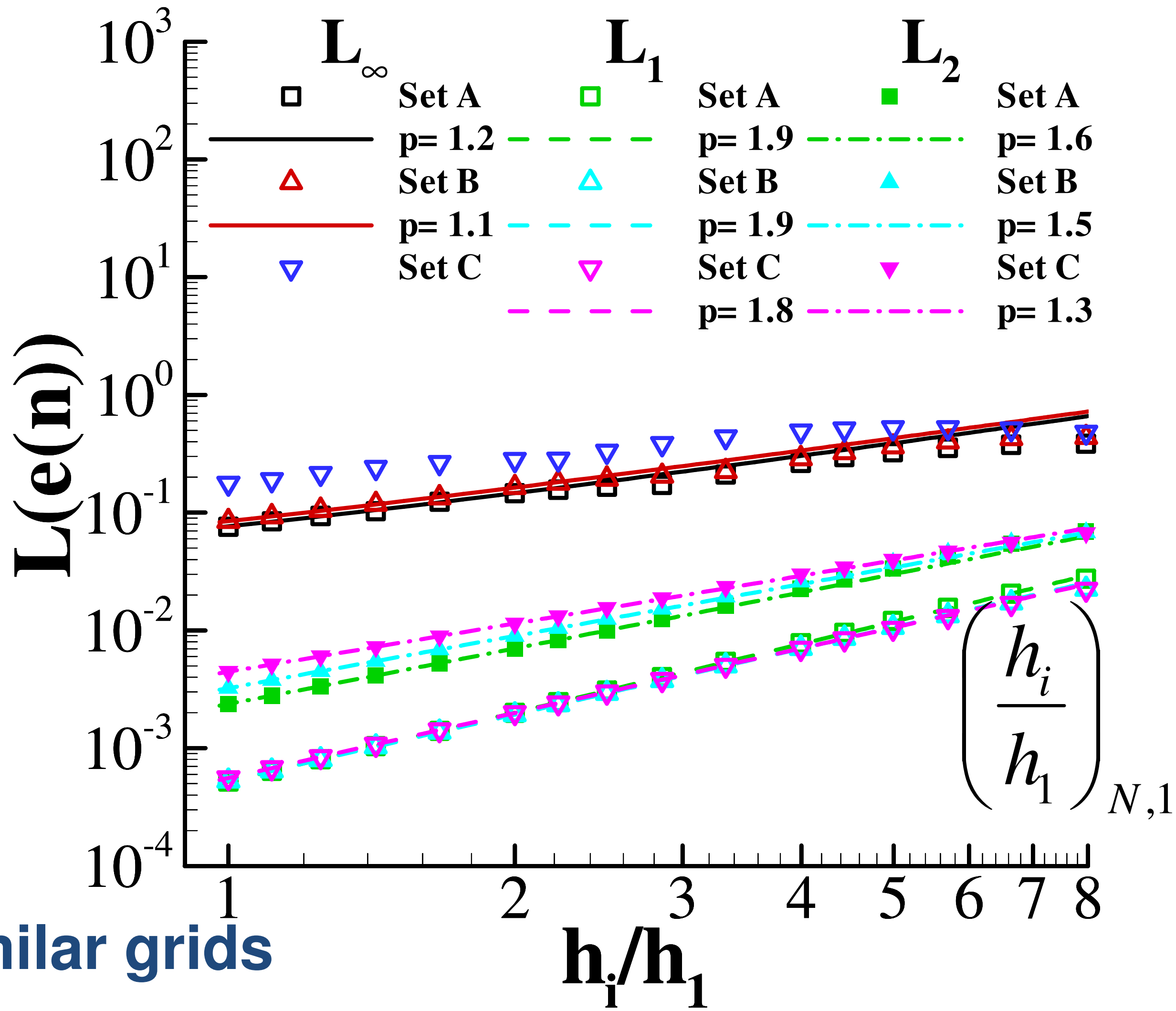

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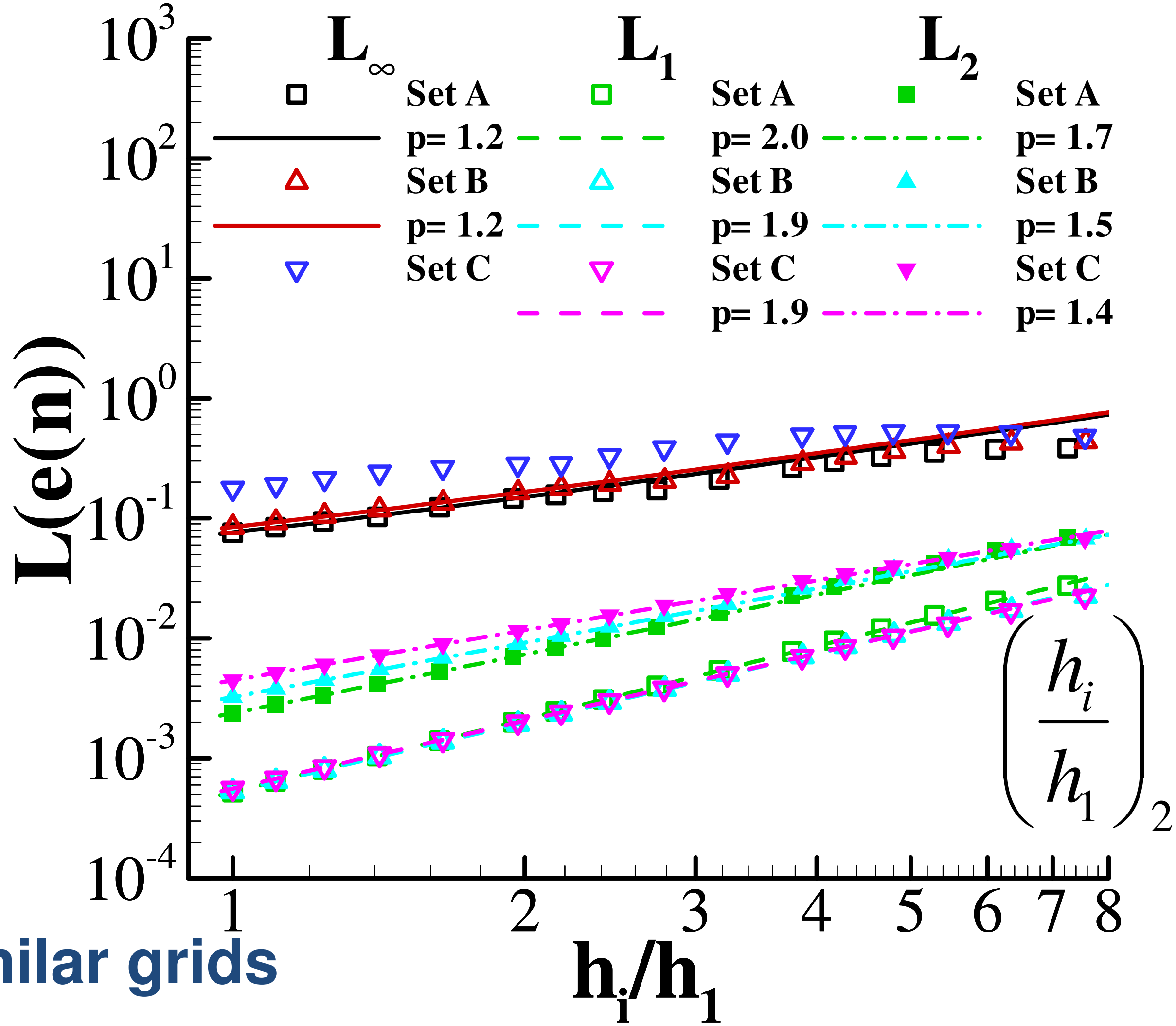

Unsimilar grids

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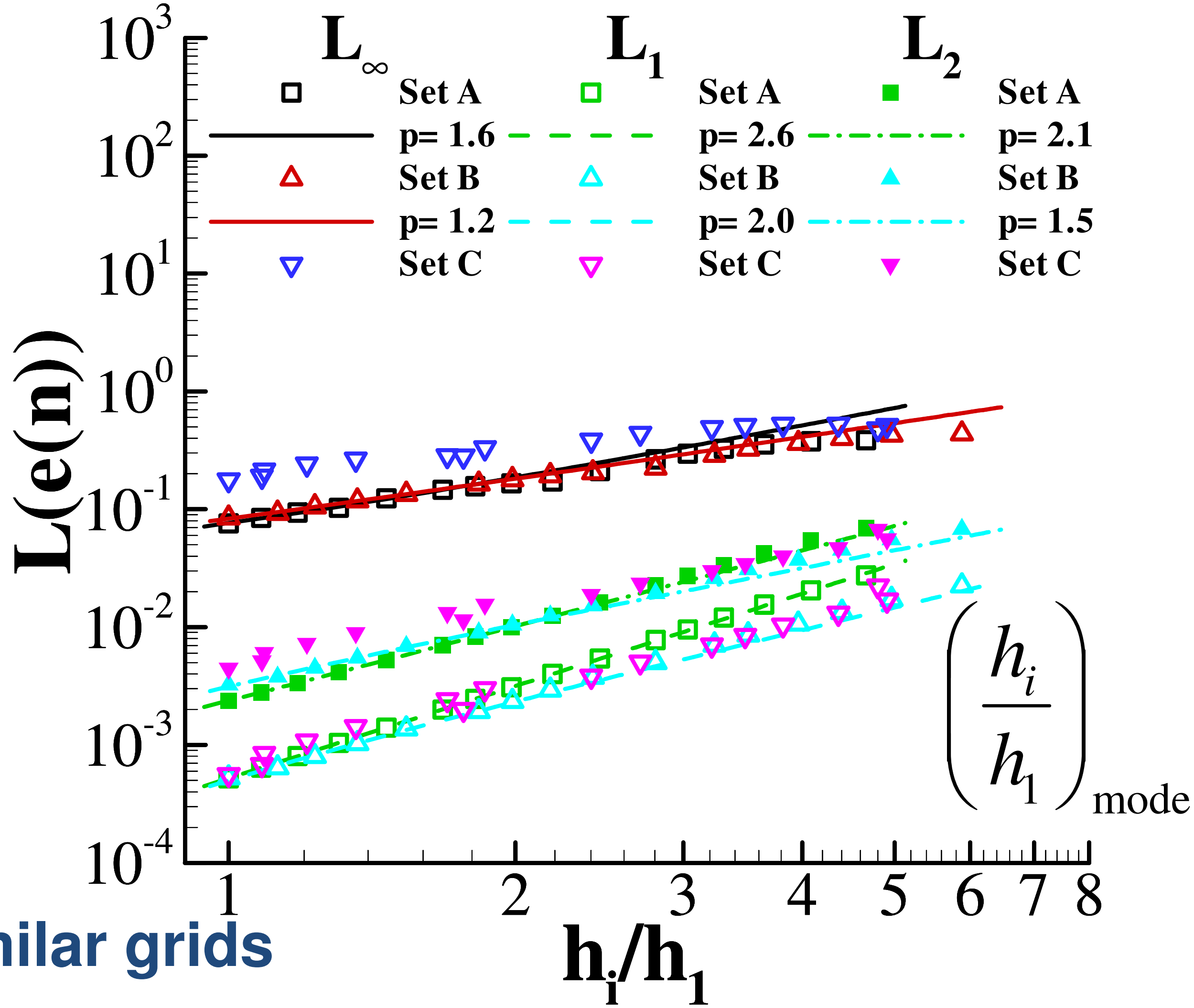

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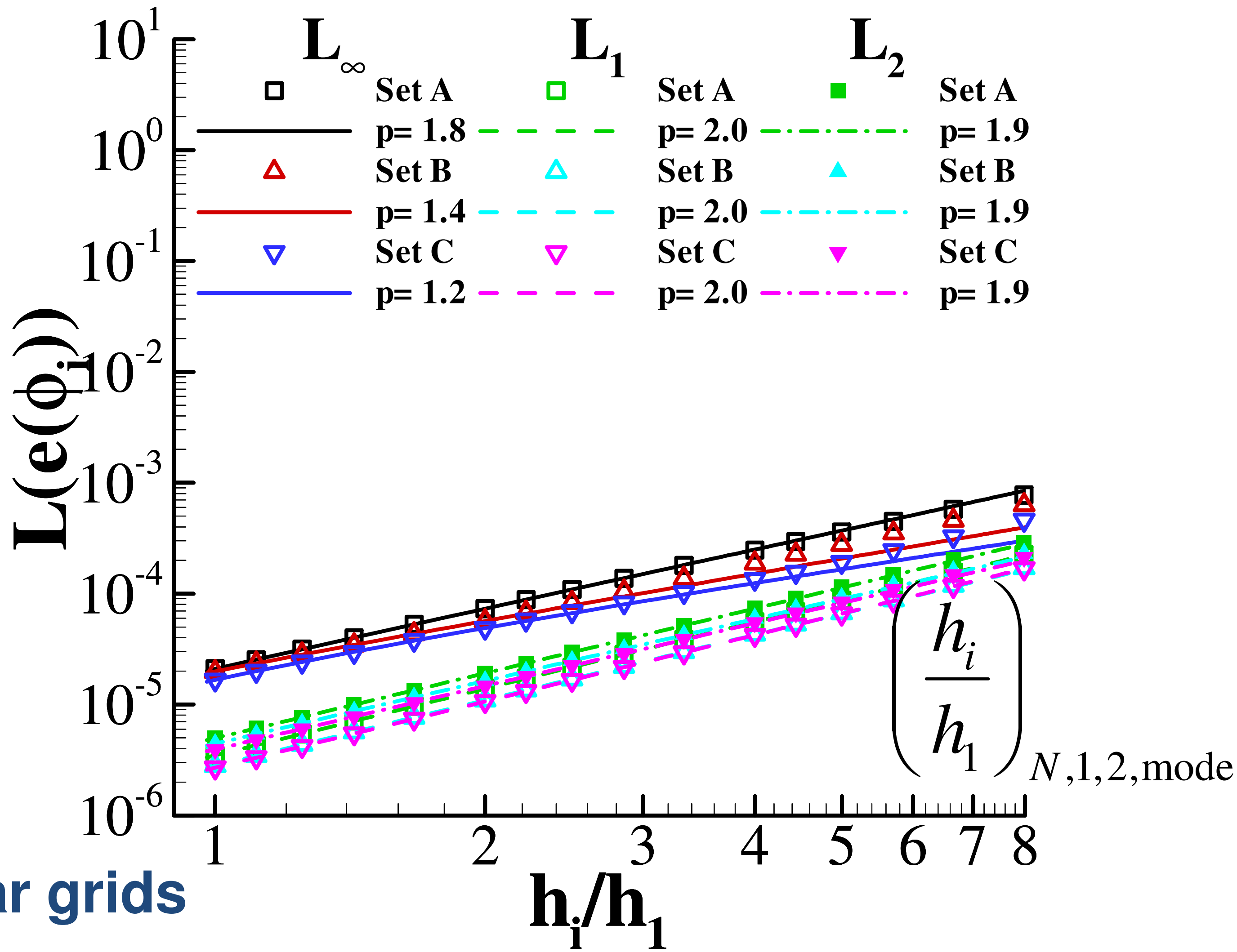

Similar grids

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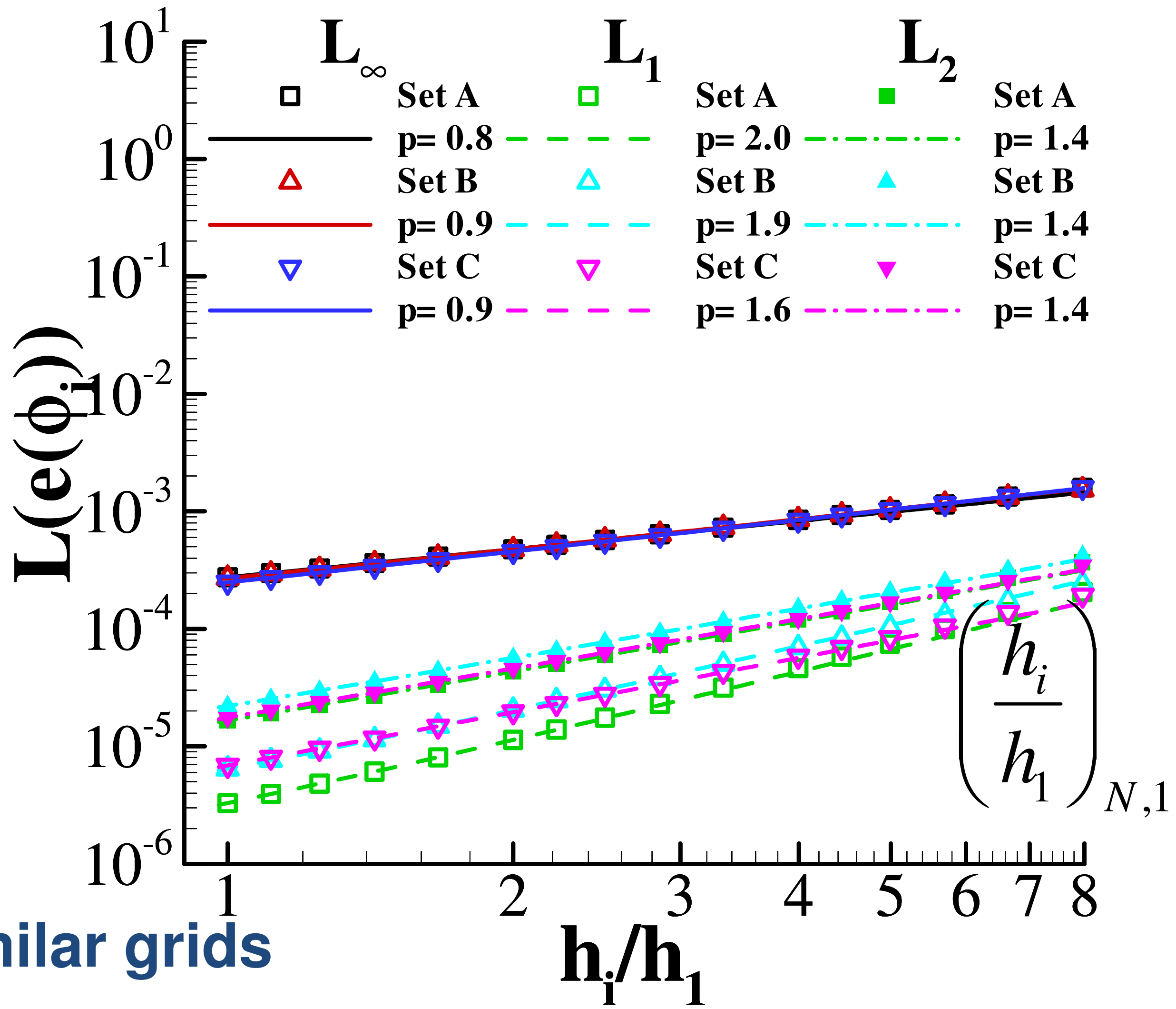

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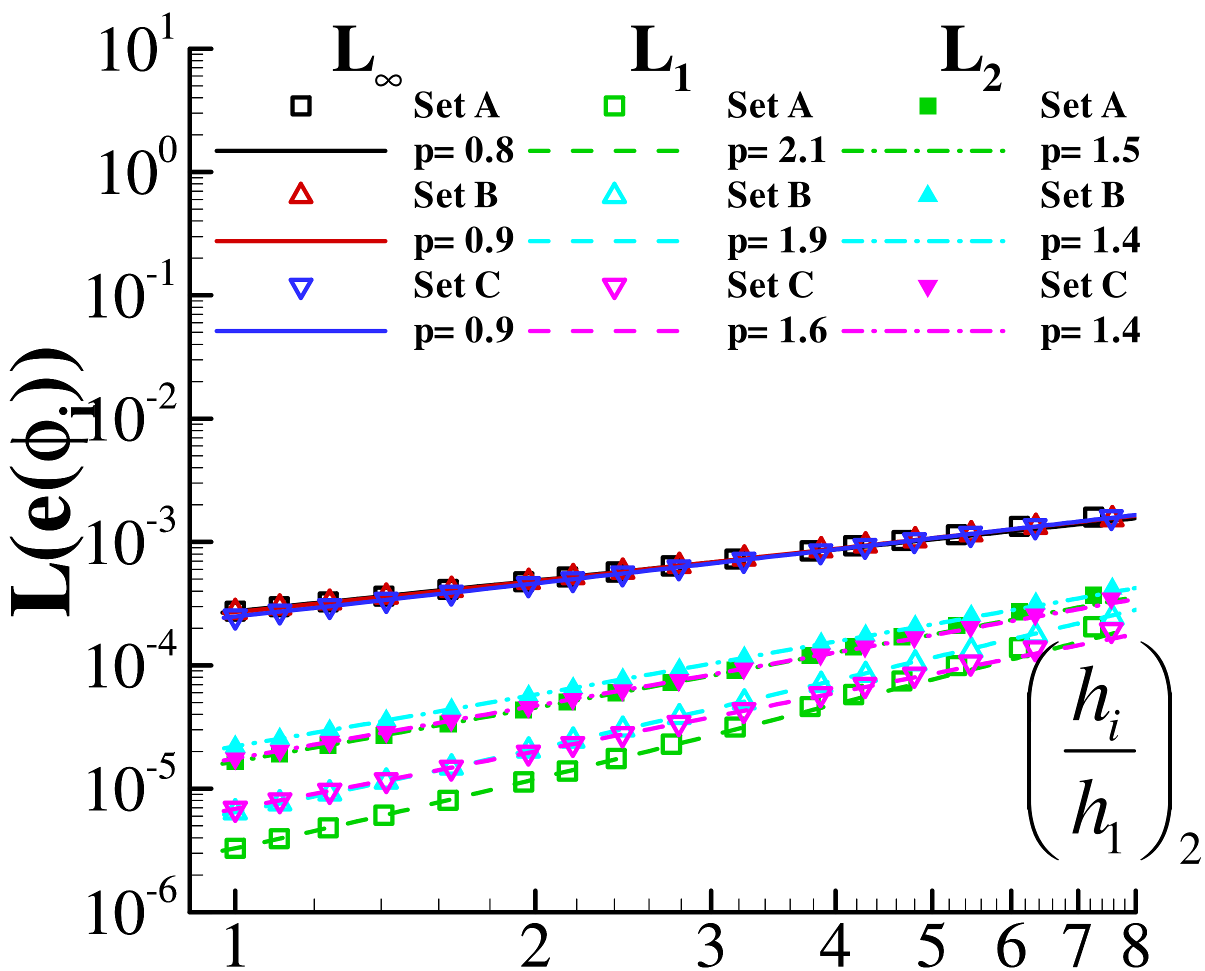

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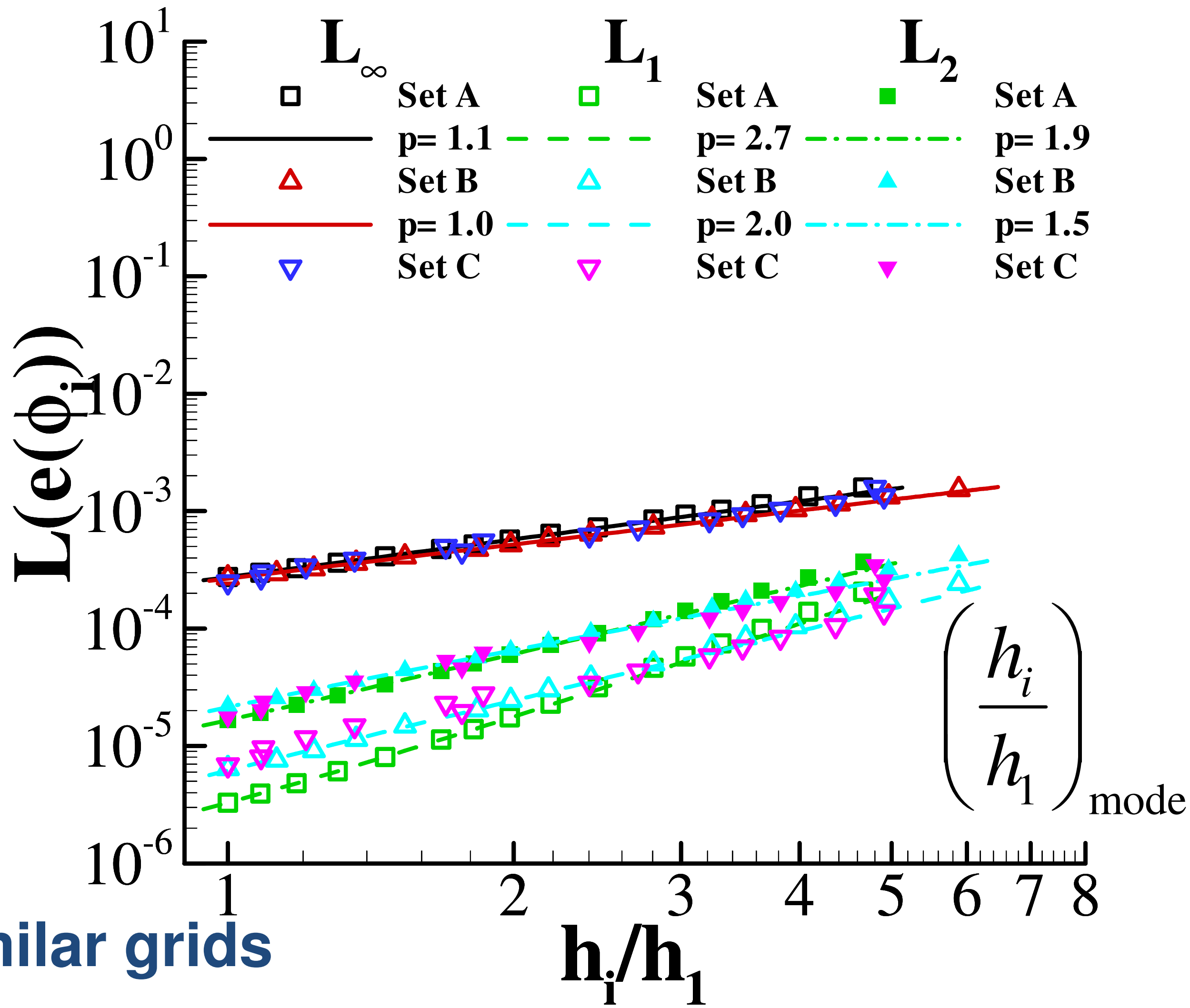

Unsimilar grids

6. Final Remarks

- Error estimation based on power series expansions does not require geometrically similar grids
- Definition of typical cell size may be based on number of cells, average cell size or root mean square of the cell size
- Mode of the cell size requires convergence of the statistics and may lead to awkward results
- Observed order of accuracy is only reliable for geometrically similar grid sets (where any consistent definition of the typical cell size leads to the same result)